Potential Vorticity (PV) Dynamics and Models of Zonal Flow Formation

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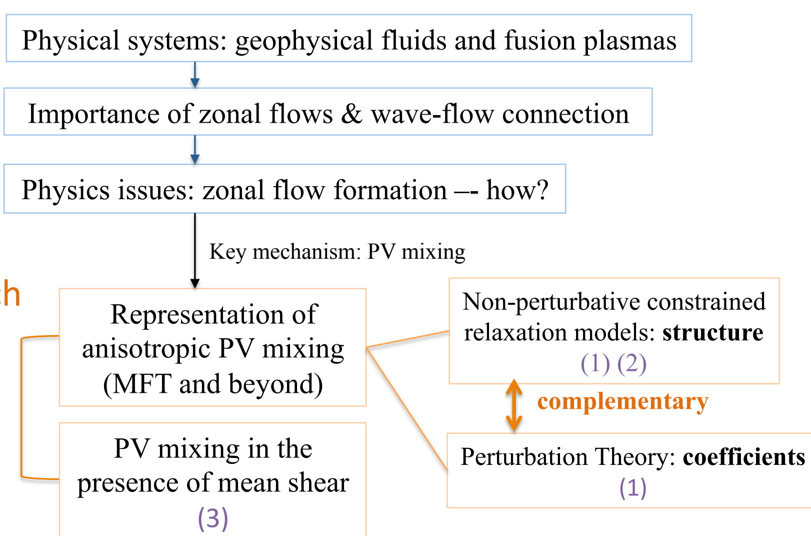
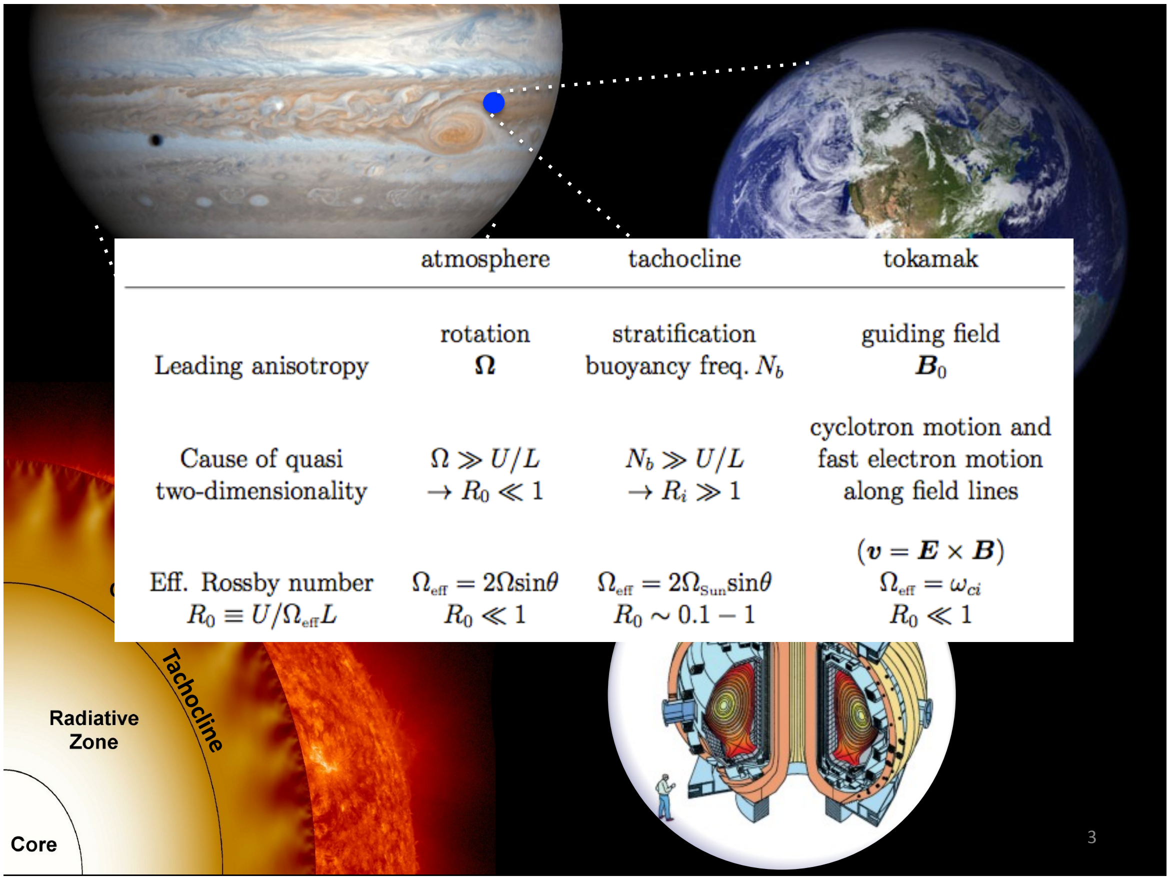
Outline

- Introduction
- Thesis research



- Summary : What did we learn?

(1) Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 032314 (2015)
(2) Pei-Chun Hsu, P. H. Diamond, and S. M. Tobias, Phys. Rev. E, in press (2015)
(3) Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 022306, (2015)



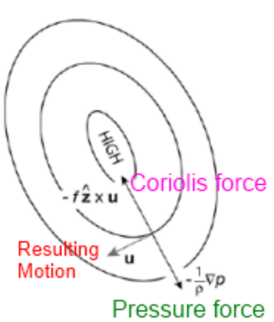
	atmosphere	tachocline	tokamak
Leading anisotropy	rotation $\boldsymbol{\Omega}$	stratification buoyancy freq. N_b	guiding field $\boldsymbol{B}_0$
Cause of quasi two-dimensionality	$\Omega \gg U/L$ $\rightarrow R_0 \ll 1$	$N_b \gg U/L$ $\rightarrow R_i \gg 1$	cyclotron motion and fast electron motion along field lines
Eff. Rossby number $R_0 \equiv U/\Omega_{\text{eff}}L$	$\Omega_{\text{eff}} = 2\Omega\sin\theta$ $R_0 \ll 1$	$\Omega_{\text{eff}} = 2\Omega_{\text{Sun}}\sin\theta$ $R_0 \sim 0.1 - 1$	$(\boldsymbol{v} = \boldsymbol{E} \times \boldsymbol{B})$ $\Omega_{\text{eff}} = \omega_{ci}$ $R_0 \ll 1$

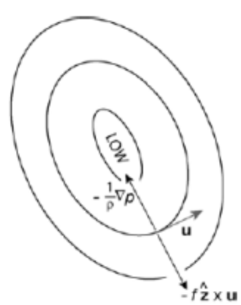
Physical systems I

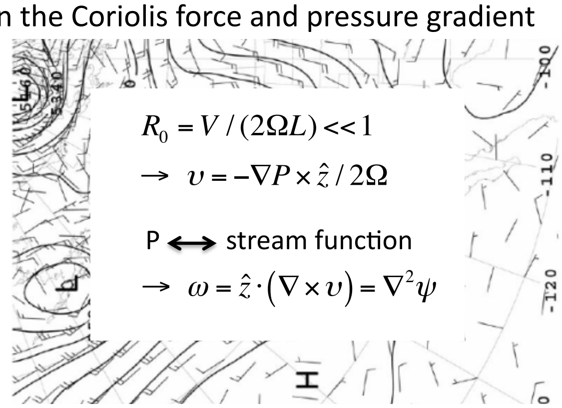
Geophysical fluids

- Phenomena: weather, waves, large scale atmospheric and oceanic circulations, water circulation, jets…
- Geophysical fluid dynamics (GFD): low frequency ($\omega < \Omega$)

  "We might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician of the Beethoven than the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand." – J.G. Charney
- Geostrophic motion: balance between the Coriolis force and pressure gradient







$R_0 = V/(2\Omega L) << 1$

$\rightarrow \quad \upsilon = -\nabla P \times \hat{z} / 2\Omega$

P ⟷ stream function

$\rightarrow \quad \omega = \hat{z} \cdot (\nabla \times \upsilon) = \nabla^2 \psi$

Physical systems II

Kelvin's theorem – unifying principle throughout

- Kelvin's circulation theorem for rotating system

$$\oint v \cdot dl = \int (\nabla \times v + 2\Omega) \cdot \hat{z}\, dS \equiv C \qquad \dot{C} = 0$$

relative planetary

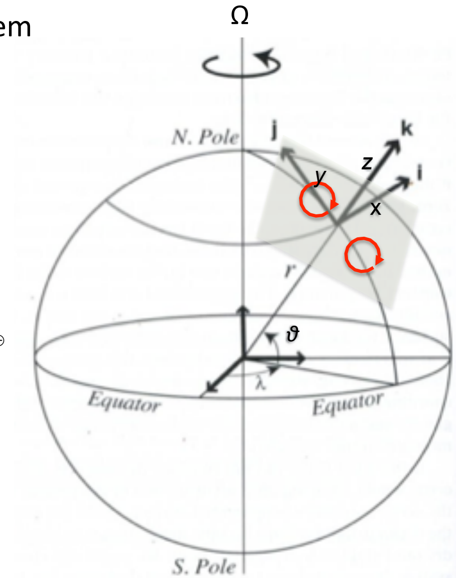
- Displacement on beta-plane

$$\dot{C} = 0 \;\rightarrow\; \frac{d}{dt}\nabla^2\psi = -2\Omega\cos\theta \frac{d\theta}{dt} = -\beta v_y$$

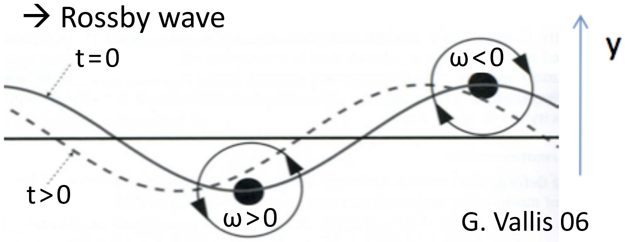
$$\beta = 2\Omega\cos\theta_0 / R_{\oplus}$$

- Quasi-geostrophic eq

$$\frac{d}{dt}\left(\nabla^2\psi + \beta y\right) = 0$$ **PV conservation**



→ Rossby wave



G. Vallis 06

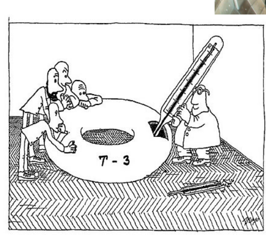
Physical systems III

Magnetically confined plasma

- Nuclear fusion: option for generating large amounts of carbon-free energy
- Challenge: ignition -- reaction release more energy than the input energy

  Lawson criterion:

$$n_i \tau_E T_i > 3 \times 10^{21} \mathrm{m}^{-3}\mathrm{s\,keV}$$



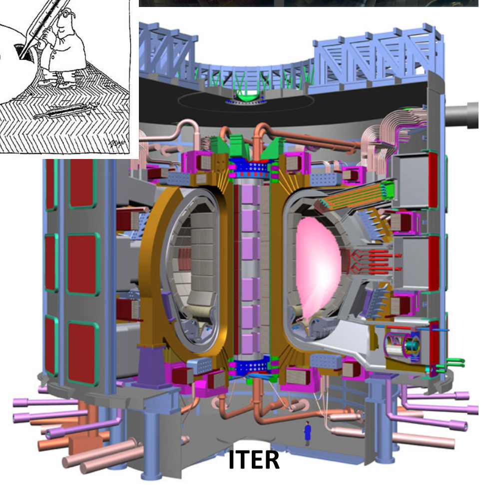
  → confinement

  → turbulent transport

- Turbulence: instabilities and collective oscillations

  → lowest frequency modes dominate the transport

  → drift wave



Physical systems IV

Drift wave model – Fundamental prototype

- Hasegawa-Wakatani : simplest model incorporating instability

$$V = \frac{c}{B}\hat{z}\times\nabla\phi + V_{pol}$$

$$J_\perp = n|e|V^i_{pol} \qquad \eta J_\parallel = -\nabla_\parallel\phi + \nabla_\parallel p_e$$

$$\nabla_\perp\cdot J_\perp + \nabla_\parallel J_\parallel = 0 \rightarrow \text{vorticity: } \rho_s^2\frac{d}{dt}\nabla^2\phi = -D_\parallel\nabla_\parallel^2(\phi - n) + \nu\nabla^2\nabla^2\phi$$

$$\frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0 \rightarrow \text{density: } \frac{d}{dt}n = -D_\parallel\nabla_\parallel^2(\phi - n) + D_0\nabla^2 n$$

→ **PV conservation in inviscid theory** $\frac{d}{dt}\left(n - \nabla^2\phi\right) = 0$

→ PV flux = particle flux + vorticity flux

→ zonal flow being a counterpart of particle flux

QL: $\frac{\partial}{\partial t}\langle n\rangle = -\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\tilde{n}\rangle$

→? $\frac{\partial}{\partial t}\langle\nabla^2\phi\rangle = -\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\nabla^2\tilde{\phi}\rangle = -\frac{\partial^2}{\partial r^2}\langle\tilde{\upsilon}_r\tilde{\upsilon}_\theta\rangle$

- Hasegawa-Mima ($D_\parallel k_\parallel^2/\omega >> 1 \rightarrow n \sim \phi$)

$$\frac{d}{dt}\left(\phi - \rho_s^2\nabla^2\phi\right) + \upsilon_*\partial_y\phi = 0$$

Physical systems: unifying concept

PV conservation

- **PV conservation dq/dt=0**

GFD: Quasi-geostrophic system	Plasma: Hasegawa-Wakatani system
$q = \nabla^2\psi + \beta y$ relative vorticity ($\nabla^2\psi$), planetary vorticity (βy)	$q = n - \nabla^2\phi$ density (guiding center) (n), ion vorticity (polarization) ($\nabla^2\phi$)
Physics: $\Delta y \rightarrow \Delta\left(\nabla^2\psi\right)$	Physics: $\Delta r \rightarrow \Delta n \rightarrow \Delta\left(\nabla^2\phi\right)$ ZF!

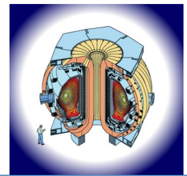
- Charney-Haswgawa-Mima equation

$$n = n_0 + \tilde{n} \qquad \tilde{n} \sim \frac{e\tilde{\phi}}{T}$$

H-W → H-M: $\frac{1}{\omega_{ci}}\frac{\partial}{\partial t}\left(\nabla^2\phi - \rho_s^{-2}\phi\right) - \frac{1}{L_n}\frac{\partial}{\partial y}\phi + \frac{\rho_s}{L_n}J(\phi,\nabla^2\phi) = 0$

Q-G: $\frac{\partial}{\partial t}\left(\nabla^2\psi - L_d^{-2}\psi\right) + \beta\frac{\partial}{\partial x}\psi + J(\psi,\nabla^2\psi) = 0$

Physical systems: summary

turbulence	quasi-geostrophic	drift-wave
→ force	Coriolis	Lorentz
velocity	geostrophic	magnetic energy
linear waves	Rossby waves	drift waves
→ conserved PV	$q = \nabla^2\psi + \beta y$	$q = n - \nabla^2\phi$
→ inhomogeneity	β	$\nabla n,\ \nabla T$
→ characteristic scale	$L_D \approx 10^6 m$	$\rho_s \approx 10^{-3} m$
→ fast frequency	$f \approx 10^{-2} s^{-1}$	$\omega_{ci} \approx 10^8 s^{-1}$
turbulence	usually strongly driven	not far from marginal
Reynolds number	R_e >>1	R_e ~ 10-10^2
→ zonal flows	Jets, zonal bands	sheared E x B flows
role of zonal flows	transport barriers	L-H transition

Outline

- Introduction
  - Physical systems: geophysical fluids and fusion plasmas
  - Key feature: zonal flows
  - **Importance of zonal flows**
    - **atmospheric phenomena**
    - **confinement of fusion plasma**
  - Physics issues: zonal flow formation
- Thesis research
  - Representation of anisotropic PV mixing (MFT and beyond)
    - Constrained relaxation models (1) (2)
    - Perturbation Theory (1)
  - PV mixing in the presence of mean shear (3)
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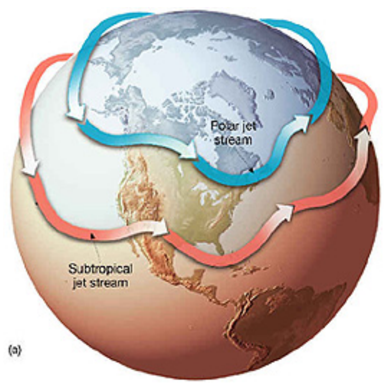
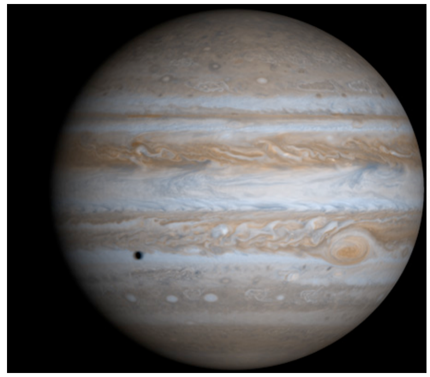
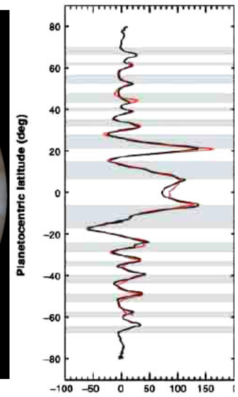
ZF importance I

Atmospheric phenomena







- Phenomena: jet streams, Jovian zonal bands and zones,… → zonal flows
- Zonal flows have a crucial influence on turbulent mixing and formation of transport barriers.
  -- inhibiting the transport of vortex eddies across the flows
  -- ozone hole problem <-> transport barrier (shear layer insulation)

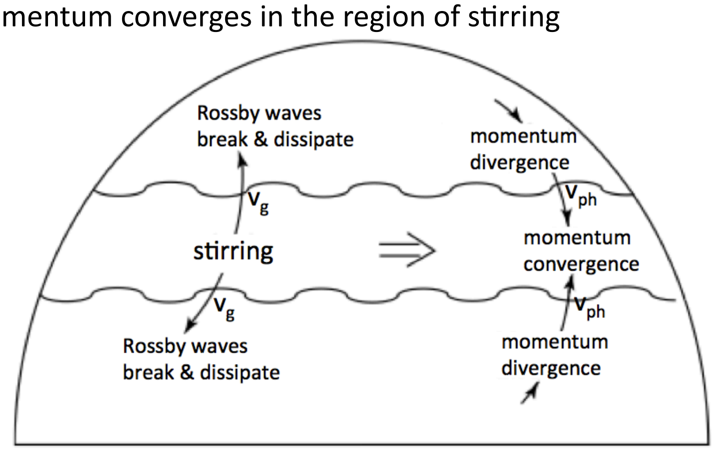
Wave-flow connection I

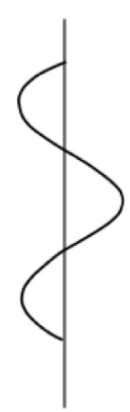
Mid-latitude circulation

- Stirring generates Rossby waves: $\omega_k = \frac{-\beta k_x}{k^2}$, $\upsilon_{gy} = 2\beta k_x k_y / k^4$, $\upsilon_{phy} = -\beta k_x / k_y k^2$ **opposite**
- Waves propagate away from the disturbance

  energy density flux: $\upsilon_{gy}(\nabla\psi_k)^2/2 = \beta k_x k_y k^{-2}|\psi_k|^2$
  eddy zonal-momentum flux: $\langle\tilde{\upsilon}_x\tilde{\upsilon}_y\rangle = -k_x k_y|\tilde{\psi}_k|^2$ **opposite**

→ Momentum converges in the region of stirring





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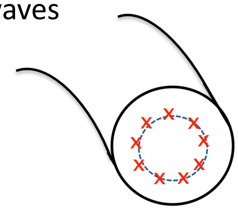
Wave-flow connection II

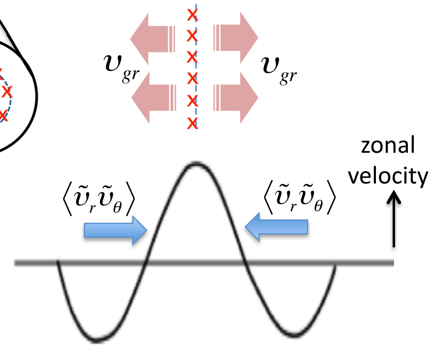
Wave radiation in DW turbulence

- Localized instability drives drift waves

$$\omega_k = \frac{k_\theta \upsilon_*}{1 + k_\perp^2 \rho_s^2}, \quad \upsilon_* < 0$$

$$\upsilon_{gr} \upsilon_{phr} < 0$$





- Outgoing wave energy flux
  → incoming wave momentum flux

$$\upsilon_{gr} \frac{(\nabla \phi_k)^2}{2} = -\rho_s^2 \frac{k_x k_y \upsilon_*}{(1 + k_\perp^2 \rho_s^2)^2} k^{-2} |\psi_k|^2$$

$$\langle \tilde{\upsilon}_r \tilde{\upsilon}_\theta \rangle = -\frac{c^2}{B^2} k_x k_y |\tilde{\phi}_k|^2$$

opposite

→ Zonal flow layers form at excitation regions

- Essential elements in zonal flow generation
  -- inhomogeneous mixing in space
  -- one direction of symmetry (nature of torus and planet)

ZF importance II

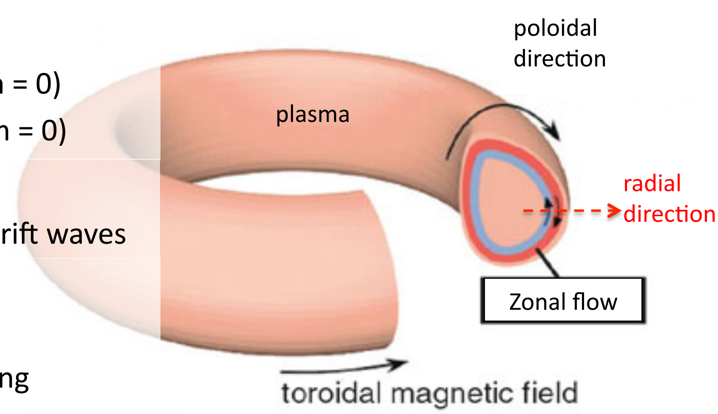
Confinement of fusion plasma

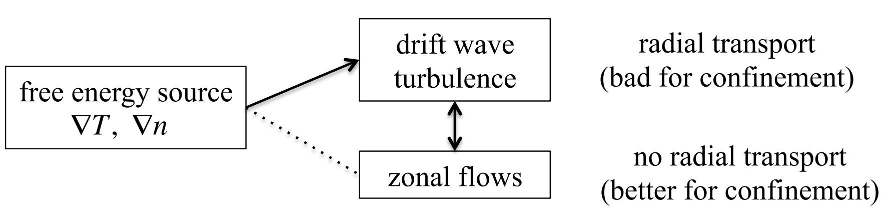
- Zonal flows:

-- sheared E×B flows
  poloidally symmetric (n = 0)
  toroidally symmetric (m = 0)
  → cannot tap free energy
  → driven nonlinearly by drift waves

-- modes of
  minimum inertia
  minimal Landau damping
  no radial transport

→ important for plasma confinement





ZF importance III

L-H transition

L-mode

- Increase of heating power -> heat flux
- edge turbulence -> Reynolds stress
- development of zonal flows

→ LCO

- regulation of turbulence and transport (self-regulation → oscillation)
- buildup of a steep pressure gradient
- growth of the mean shear

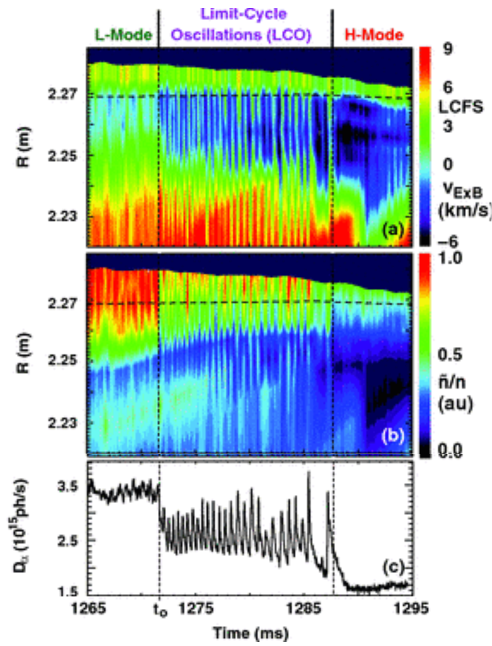
→ H-mode transition (onset: G. Tynan et al 2013)

damping of both turbulence and zonal flows by mean shear

ZF and MS play different roles

ZF: extracts from turbulence

MS: locks in transition



L. Schmitz et al 2012

Outline

- Introduction
  - Physical systems: geophysical fluids and fusion plasmas
  - Importance of zonal flows

Understanding turbulence-zonal flow interaction

Physics issues: zonal flow formation

- Thesis research
  - Representation of anisotropic PV mixing (MFT and beyond)
    - Constrained relaxation models (1) (2)
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- Summary : What did we learn?

PV mixing & Flows I

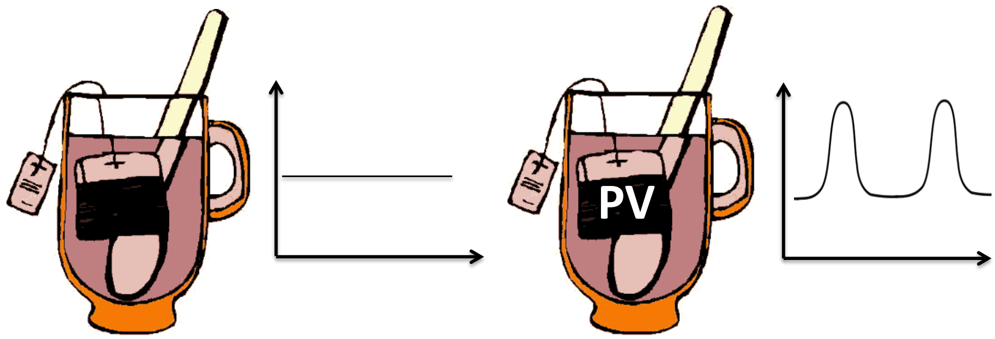
Zonal flow formation

- Zonal flows are generated by nonlinear interactions between wave turbulence and zonal flow.
- In x space, zonal flows are driven by Reynolds stress $\frac{\partial}{\partial t}\langle v_x\rangle = -\frac{\partial}{\partial y}\langle \tilde{v}_x\tilde{v}_y\rangle - \mu\langle v_x\rangle$

  Taylor's Identity $\langle \tilde{v}_y\tilde{q}\rangle = -\frac{\partial}{\partial y}\langle \tilde{v}_x\tilde{v}_y\rangle$ → **PV flux fundamental to zonal flow formation**

- Inhomogeneous PV mixing, not momentum mixing (dq/dt=0)

  → up-gradient momentum transport (negative-viscosity) not an enigma

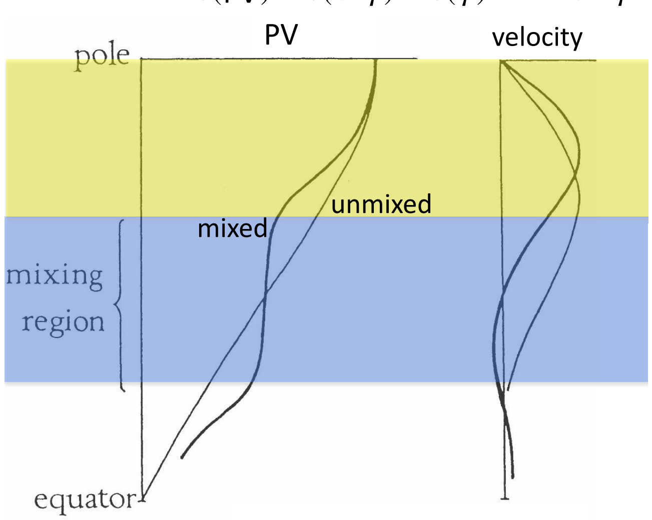


PV mixing & Flows II

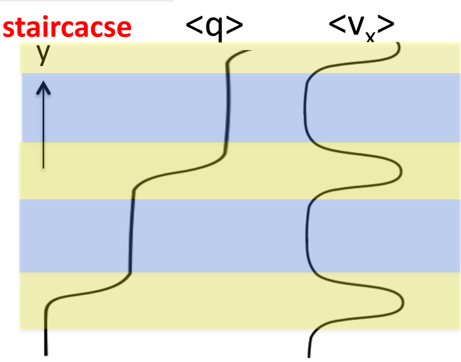
Inhomogeneous PV mixing

- **PV mixing is the fundamental mechanism for zonal flow formation** → **PV staircacse**

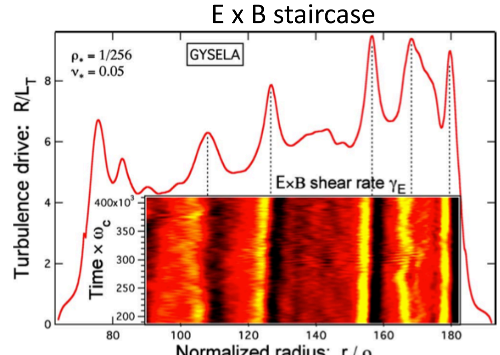
$$\delta(\mathrm{PV}) \rightarrow \delta(\nabla^2\psi) \rightarrow \delta(\psi) \rightarrow v = \nabla\times\psi$$



McIntyre 1982



E x B staircase



G. Dif-Pradalier et al 2010

Outline

- Introduction

Physical systems: geophysical fluids and fusion plasmas

Importance of zonal flows

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Key mechanism: PV mixing

- Thesis research

Representation of anisotropic PV mixing (MFT and beyond)

Non-perturbative constrained relaxation models: structure (1) (2)

complementary

Perturbation theory: coefficients (1)

PV mixing in the presence of mean shear (3)

- Summary

(1) Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 032314 (2015)
(2) Pei-Chun Hsu, P. H. Diamond, and S. M. Tobias, Phys. Rev. E, in press (2015)
(3) Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 022306, (2015)

PV flux I
General Structure

Non-perturbative approaches

– PV mixing in space is essential in ZF generation.

Taylor identity: $\left\langle \tilde{v}_y \nabla^2 \tilde{\phi} \right\rangle = -\partial_y \left\langle \tilde{v}_y \tilde{v}_x \right\rangle$

vorticity flux   Reynolds force

Key:
How represent inhomogeneous PV mixing

General structure of PV flux?
→relaxation principles!

most treatment of ZF:
-- perturbation theory
-- modulational instability
(test shear + gas of waves)
~ linear theory

-> physics of evolved PV mixing?
-> something more general?

non-perturb model 1: use selective decay principle

What form must the PV flux have so as to dissipate enstrophy while conserving energy?

non-perturb model 2: use joint reflection symmetry

What form must the PV flux have so as to satisfy the joint reflection symmetry principle for PV transport/mixing?

non-perturb model 1

General principle: selective decay

- 2D turbulence conservation of energy and potential enstrophy

→ dual cascade

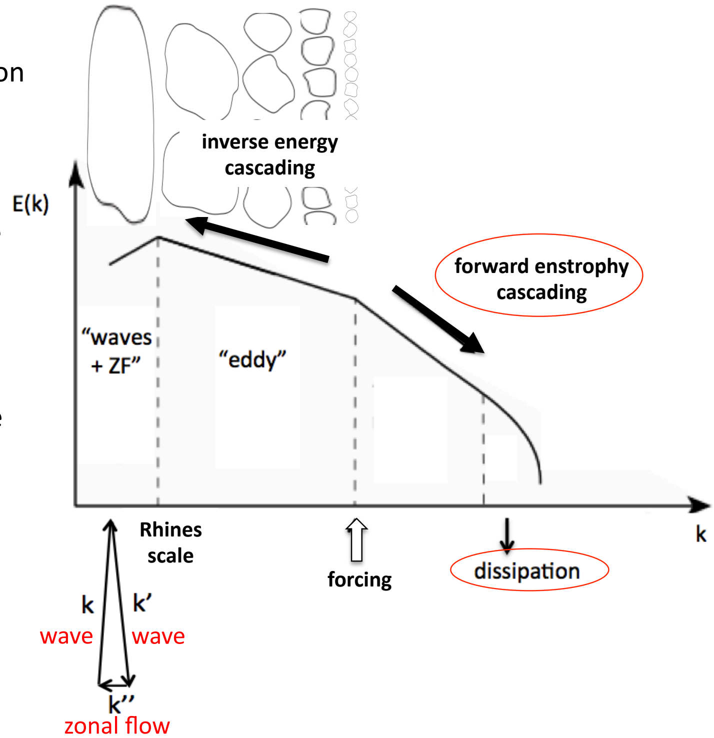
→ Minimum enstrophy state

- eddy turnover rate and Rossby wave frequency mismatch are comparable

$$\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega + \beta \upsilon = 0$$

$$\frac{U}{LT} \quad \frac{U^2}{L^2} \quad \beta U$$

→ Rhines scale $L_R \sim \sqrt{\frac{U}{\beta}}$



non-perturb model 1

Using selective decay for flux

	minimum enstrophy relaxation (Bretherton & Haidvogel 1976)	analogy ⟷	Taylor relaxation (J.B. Taylor, 1974)
turbulence	2D hydro		3D MHD
conserved quantity (constraint)	total kinetic energy		global magnetic helicity
dissipated quantity (minimized)	fluctuation potential enstrophy		magnetic energy
final state	minimum enstrophy state flow structure emergent		Taylor state force free B field configuration
structural approach	$\frac{\partial}{\partial t}\Omega < 0 \Rightarrow \Gamma_E \Rightarrow \Gamma_q$		$\frac{\partial}{\partial t}E_M < 0 \Rightarrow \Gamma_H$

dual cascade (conserved quantity, dissipated quantity)

- flux? what can be said about dynamics?

→ structural approach (this work): *What form must the PV flux have so as to dissipate enstrophy while conserving energy?*

General principle based on general physical ideas → useful for dynamical model

non-perturb model 1

PV flux

→ PV conservation

mean field PV: $$\frac{\partial\langle q\rangle}{\partial t}+\partial_y\langle \upsilon_y q\rangle=\nu_0\partial_y^2\langle q\rangle$$

Γ_q : mean field PV flux

Key Point: what form does PV flux have s/t dissipate enstrophy, conserve energy

selective decay

→ energy conserved $E=\int\frac{\left(\partial_y\langle\phi\rangle\right)^2}{2}$

$$\frac{\partial E}{\partial t}=\int\langle\phi\rangle\partial_y\Gamma_q=-\int\partial_y\langle\phi\rangle\Gamma_q \qquad \Rightarrow\Gamma_q=\frac{\partial_y\Gamma_E}{\partial_y\langle\phi\rangle}$$

→ enstrophy minimized $\Omega=\int\frac{\langle q\rangle^2}{2}$

$$\frac{\partial\Omega}{\partial t}=-\int\langle q\rangle\partial_y\Gamma_q=-\int\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\Gamma_E$$

$$\frac{\partial\Omega}{\partial t}<0\Rightarrow\Gamma_E=\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right) \qquad \Rightarrow\Gamma_q=\frac{1}{\partial_y\langle\phi\rangle}\partial_y\left[\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\right]$$

general form of PV flux

μ: parameter TBD; $\partial_y\langle\phi\rangle$ → $\langle\upsilon_x\rangle$

non-perturb model 1

Structure of PV flux

$$\Gamma_q=\frac{1}{\langle\upsilon_x\rangle}\partial_y\left[\mu\,\partial_y\left(\frac{\partial_y\langle q\rangle}{\langle\upsilon_x\rangle}\right)\right] = \frac{1}{\langle\upsilon_x\rangle}\partial_y\left[\mu\left(\frac{\langle q\rangle\partial_y\langle q\rangle}{\langle\upsilon_x\rangle^2}+\frac{\partial_y^2\langle q\rangle}{\langle\upsilon_x\rangle}\right)\right]$$

μ: diffusion parameter calculated by perturbation theory, numerics…

diffusion and hyper diffusion of PV

<--> usual story : Fick's diffusion

relaxed state:

Homogenization of $\frac{\partial_y\langle q\rangle}{\langle\upsilon_x\rangle}$ → allows staircase

characteristic scale $\ell_c\equiv\sqrt{\left|\frac{\langle\upsilon_x\rangle}{\partial_y\langle q\rangle}\right|}$

$\ell>\ell_c$: zonal flow growth

$\ell<\ell_c$: zonal flow damping (hyper viscosity-dominated)

↔

Rhines scale $L_R\sim\sqrt{\frac{U}{\beta}}$

$\ell>L_R$: wave-dominated

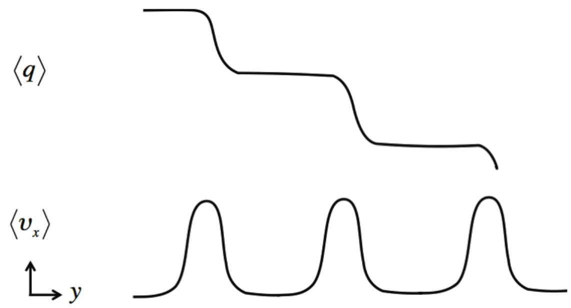
$\ell<L_R$: eddy-dominated

non-perturb model 1

PV staircase

relaxed state: homogenization of $\frac{\partial_y \langle q \rangle}{\langle v_x \rangle}$ → PV gradient large where zonal flow large

→ Zonal flows track the PV gradient → PV staircase



- Highly structured profile of the staircase is reconciled with the homogenization or mixing process required to produce it.
- Staircase may arise naturally as a consequence of minimum enstrophy relaxation.

non-perturb model 1

What sets the "minimum enstrophy"

- Decay drives relaxation. The relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state

$$\langle q \rangle = q_m(y) + \delta q(y,t)$$
$$\langle \phi \rangle = \phi_m(y) + \delta\phi(y,t)$$
$$\partial_y q_m = \lambda \partial_y \phi_m$$
$$\delta q(y,t) = \delta q_0 \exp(-\gamma_{rel} t - i\omega t + iky)$$

$$\gamma_{rel} = \mu\left(\frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle v_x \rangle^2} - \frac{8q_m^2(k^2+\lambda)}{\langle v_x \rangle^4}\right)$$
$$\omega_k = \mu\left(-\frac{4q_m k^3 + 10 q_m k\lambda}{\langle v_x \rangle^3} - \frac{8q_m^3 k}{\langle v_x \rangle^5}\right)$$

- The condition of relaxation (modes are damped):

$\Upsilon_{rel} > 0$ → $k^2 > \frac{8q_m^2}{\langle v_x \rangle^2} - 3\lambda$

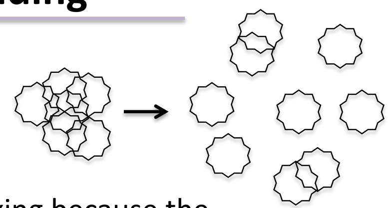
$k^2 > 0$ → $\frac{8q_m^2}{\langle v_x \rangle^2} > 3\lambda$ → Relates q_m^2 with ZF and scale factor

-- ZF cannot grow arbitrarily large, and is constrained by the enstrophy
-- To sustain a zonal flow in the minimum enstrophy state, a critical residual enstrophy density is needed.

→ q_m^2 : the 'minimum enstrophy' of relaxation, related to scale

non-perturb model 1

Role of turbulence spreading



- Turbulence spreading: tendency of turbulence to self-scatter and entrain stable regime
- Turbulence spreading is closely related to PV mixing because the transport/mixing of turbulence intensity has influence on Reynolds stresses and so on flow dynamics.
- PV mixing is related to turbulence spreading

$$\frac{\partial E}{\partial t} = \int \langle \phi \rangle \partial_y \Gamma_q = -\int \partial_y \langle \phi \rangle \Gamma_q \qquad \Rightarrow \Gamma_q = \frac{\partial_y \Gamma_E}{\partial_y \langle \phi \rangle}$$

- The effective spreading flux of turbulence kinetic energy

$$\Gamma_E = -\int \Gamma_q \langle v_x \rangle dy = -\int \frac{1}{\langle v_x \rangle} \partial_y \left[\mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right) \right] \langle v_x \rangle dy = \mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right)$$

→the gradient of the $\partial_y \langle q \rangle / \langle v_x \rangle$, drives spreading

→ the spreading flux vanishes when $\partial_y \langle q \rangle / \langle v_x \rangle$ is homogenized

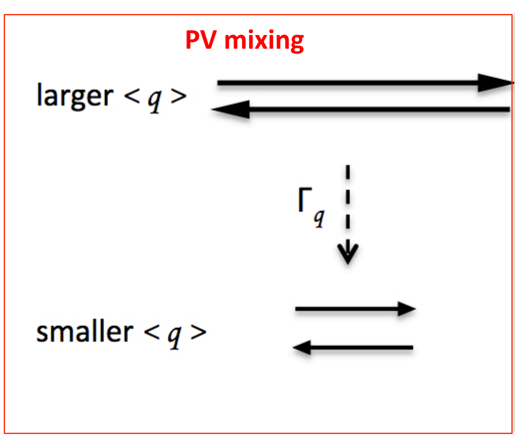
non-perturb model 1

Discussion

- PV mixing ↔ forward enstrophy cascade ↔ hyper-viscosity

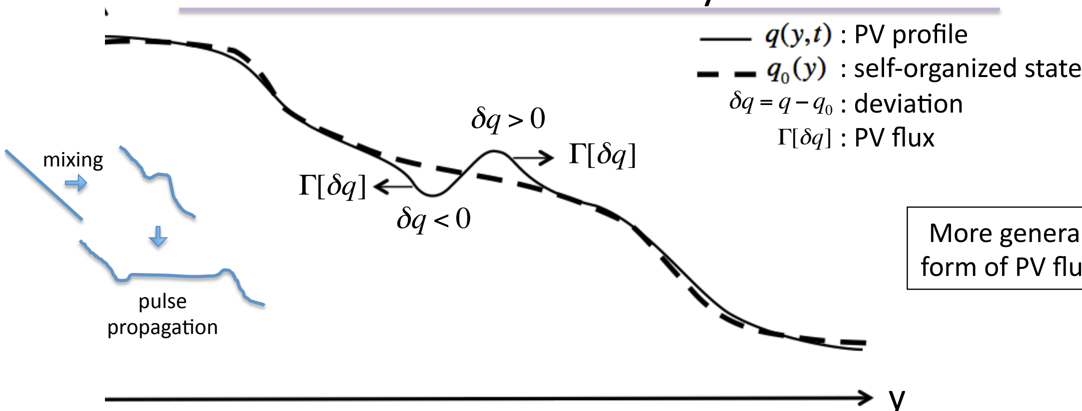
  → How to reconcile effective negative viscosity with the picture of diffusive mixing of PV in real space?

- A possible explanation of up-gradient transport of PV due to turbulence spreading



non-perturb model 2

PV-avalanche model – beyond diffusion



More general form of PV flux

- avalanching: tendency of excitation to propagate in space via local gradient change
- Joint-reflection symmetry: Γ [δq] invariant under y → -y and δq → -δq

Key Point: what form does PV flux have s/t satisfy joint-reflection symmetry principle

$$\rightarrow \Gamma[\delta q] = \sum_l \alpha_l (\delta q)^{2l} + \sum_m \beta_m (\partial_y \delta q)^m + \sum_n \gamma_n (\partial_y^3 \delta q)^n + \ldots$$

- large-scale properties : higher-order derivatives neglected
  small deviations : higher-order terms in δq neglected

  → Simplest approximation: $\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q$

non-perturb model 2

$$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q$$

- PV equation: $\partial_t \delta q + \alpha \delta q \partial_y \delta q + \beta \partial_y^2 \delta q + \gamma \partial_y^4 \delta q = 0$

Kuramoto-Sivashinsky type equation

diffusion and hyper diffusion of δq

Non-linear convection of δq

- Avalanche-like transport is triggered by deviation of PV gradient
  - → PV deviation implicitly related to the local PV gradient $\delta q \rightarrow \partial_y q$
  - → transport coefficients (functions of δq) related to the gradient $D(\delta q) \rightarrow D(\partial_y q)$
  - → gradient-dependent effective diffusion $\Gamma_q \sim -D(\partial_y q)\partial_y q \rightarrow -D(\delta q)\delta q$
- Convective component of the PV flux can be related to a gradient-dependent effective diffusivity

$$\Gamma[\delta q] \sim \delta q^2 \rightarrow -D(\delta q)\delta q$$
$$D(\delta q) \rightarrow D_0 \delta q$$

PV flux II
Setting the Coefficients

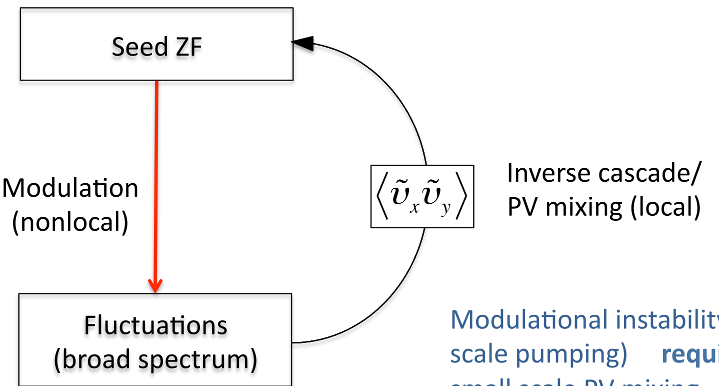
Perturbation theory

General structure of PV flux
→relaxation principles

→

Transport coefficients
→perturbation theory
(only analytical solution)

- The evolution of perturbation (seed ZF) as a way to look at PV transport



Modulational instability (large scale pumping) **requires** small scale PV mixing

perturbation theory 1
– broadband

Revisiting modulational instability

ZF evolution determined by Reynolds force

$$\frac{\partial}{\partial t}\delta V_x = -\underbrace{\frac{\partial}{\partial y}\langle \tilde{v}_x \tilde{v}_y \rangle}_{\text{vorticity flux}} = \frac{\partial}{\partial y}\sum_{k<} \frac{k_x k_y}{k^4}\tilde{N}_k$$

$N_k = k^2|\psi_k|^2/\omega_k$ is wave action density, for Rossby wave and drift wave, it is proportional to the enstrophy density. N_k is determined by WKE:

$$\frac{\partial \tilde{N}}{\partial t} + v_g \cdot \nabla \tilde{N} + \delta\omega_k \tilde{N} = \frac{\partial (k_x \delta V_x)}{\partial y}\frac{\partial N_0}{\partial k_y}$$

➜ Turbulent vorticity flux derived

$$\frac{\partial}{\partial t}\delta V_q = \partial_y^2 \delta V_q \underbrace{\sum_k \left(\frac{k_x^2 k_y}{k^4}\right)\frac{\delta\omega_k}{(\omega_q - q\cdot v_g)^2 + \delta\omega_k^2}\frac{\partial N_0}{\partial k_y}}_{\kappa(q)}$$

q: ZF wavenumber

$$\frac{\partial}{\partial t}\delta V_q = \partial_y^2 \kappa(q)\ \delta V_q$$

$\kappa(q) \neq$ const

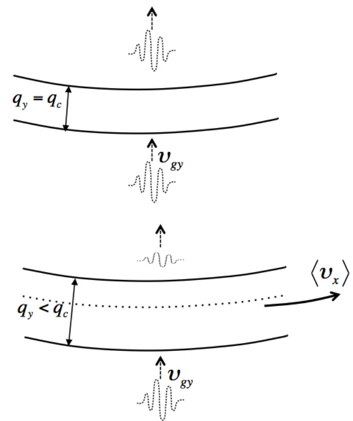
➜ scale dependence of PV flux

➜ non-Fickian turbulent PV flux

perturbation theory 1

- A simple model from which to view *κ (q)*:
  - Defining MFP of wave packets as the critical scale $q_c^{-1} \equiv \upsilon_g \delta\omega_k^{-1}$
  - keeping next order term in expansion of response

$$q^{-1} >> q_c^{-1} \Rightarrow \frac{\delta\omega_k}{(q\upsilon_g)^2 + \delta\omega_k^2} \approx \frac{1}{\delta\omega_k}\left(1 - \frac{q^2}{q_c^2}\right)$$



➜ zonal growth evolution:

$$\partial_t \delta V_x = D\partial_y^2 \delta V_x - H\partial_y^4 \delta V_x$$

➜ negative viscosity and positive hyper-viscosity

$$D = \sum_k \frac{k_x^2}{\delta\omega_k k^4} \frac{k_y \partial N_0}{\partial k_y} < 0$$

$$H = -\sum_k q_c^{-2} \frac{k_x^2}{\delta\omega_k k^4} \frac{k_y \partial N_0}{\partial k_y} > 0$$

< 0

⟺ Transport coefficients (viscosity and hyper-viscosity) for relaxation models:

$$\frac{\partial \langle \upsilon_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(-\frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$

$$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q,$$

perturbation theory 1

Discussion of *D* and *H*

- Roles of negative viscosity and positive hyper-viscosity **(Real space)**

$$\frac{\partial}{\partial t} \delta V_x = D\partial_y^2 \delta V_x - H\partial_y^4 \delta V_x$$

$$\frac{\partial}{\partial t} \int \frac{1}{2} \delta V_x^2 d^2x = -D \int (\partial_y \delta V_x)^2 d^2x - H \int (\partial_y^2 \delta V_x)^2 d^2x$$

$D < 0 \Rightarrow \gamma_{q,D} > 0$	ZF growth (Pumper D)
$H > 0 \Rightarrow \gamma_{q,H} < 0$	ZF suppression (Damper H)

⇨ Energy transferred to large scale ZF

➜ D, H as model of spatial PV flux beyond over-simplified negative viscosity

$D = Hq^2$ sets the cut-off scale

$$\Rightarrow l_c^2 = \sqrt{\frac{H}{|D|}}$$

⟺ **Minimum enstrophy model**

$$\Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \, \partial_y \left(\frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right] \Rightarrow \ell_c \equiv \sqrt{\left| \frac{\langle \upsilon_x \rangle}{\partial_y \langle q \rangle} \right|}$$

$\ell > \ell_c$: ZF energy growth → D process dominates at large scale

$\ell < \ell_c$: ZF energy damping → H process dominates at small scale perturbation theory 2 – narrowband

Parametric instability

	pseudo-fluid	plasma fluid
elements	wave-packets	charged particles (species α)
distribution function	$N_{\boldsymbol{k}}(\boldsymbol{k}, \omega_{\boldsymbol{k}})$	$f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$
mean free path	$\lvert \boldsymbol{v}_g \rvert / \delta\omega_{\boldsymbol{k}}$	$1/n_\alpha \sigma$
density	$n^w = \int N_{\boldsymbol{k}} d\boldsymbol{k}$	$n_\alpha = \int f_\alpha d\boldsymbol{v}$
momentum	$\boldsymbol{P}^w = \int \boldsymbol{k} N_{\boldsymbol{k}} d\boldsymbol{k}$	$\boldsymbol{p}_\alpha = \int m_\alpha \boldsymbol{v} f_\alpha d\boldsymbol{v}$
velocity	$\boldsymbol{V}^w = \frac{\int \boldsymbol{v}_g N_{\boldsymbol{k}} d\boldsymbol{k}}{\int N_{\boldsymbol{k}} d\boldsymbol{k}}$	$\boldsymbol{u}_\alpha = \frac{\int \boldsymbol{v} f_\alpha d\boldsymbol{v}}{\int f_\alpha d\boldsymbol{v}} = \frac{\boldsymbol{p}_\alpha}{m_\alpha n_\alpha}$

Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 032314 (2015)

perturbation theory 2

-- pseudo-fluid evolution:
multiplying the WKE by v_{gy} and integrating over k
normalizing by pseudo-density n^ω

$$\Rightarrow \frac{\partial}{\partial t} V_y^w + V_y^w \frac{\partial}{\partial y} V_y^w = -a \langle v_x \rangle'$$

inviscid Burgers'eq.
source: zonal shear

$$a = \int \frac{2\beta k_x^2}{k^4} \left(1 - \frac{4k_y^2}{k^2}\right) N_k dk \Big/ \int N_k \, dk$$

-- ZF evolution:

$$\langle \tilde{v}_x \tilde{v}_y \rangle = \int v_{gy} k_x N_k d^2k \cong V_y^w P_x^w$$

$$\frac{\partial}{\partial t} \langle v_x \rangle = -\frac{\partial}{\partial y} V_y^w P_x^w$$

- ZF growth rate in monochromatic limit:
  linearizing the above two eqs.

$$\gamma_q = \sqrt{q^2 k_x^2 |\varphi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right)}$$

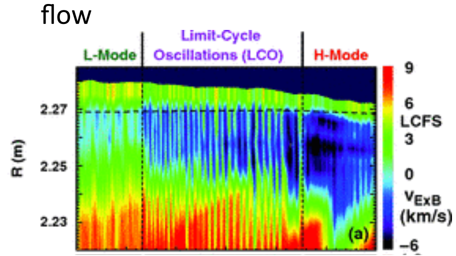
The reality of γ_q requires $k_x^2 > 3k_y^2$
$\gamma_q \propto |q|$ indicates convective instability

	PV flux	convective	viscous	hyper-viscous	coefficients
(non-perturb.)	Min. enstrophy relaxation		•	•	
	PV-avalanche relaxation	•	•	•	
(perturbative)	Modulational instability		•	•	$D_t(<0), H_t(>0)$
	Parametric instability	•			$\gamma_q(\sim \lvert q \rvert)$

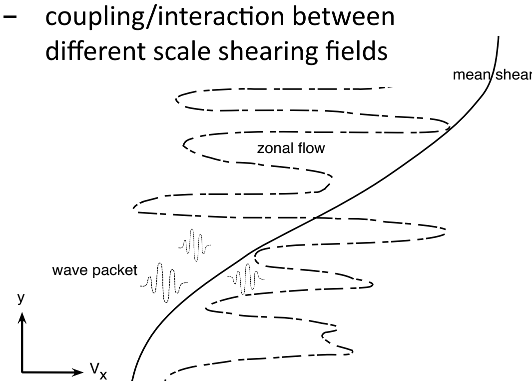
- Minimum enstrophy $$\frac{\partial \langle v_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(-\frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$
- PV-avalanche $$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q$$
- Modulational instab. $$\partial_t \delta V_x = -q^2 D \delta V_x + q^4 H \delta V_x$$
- Parametric instab. $$\gamma_q = \sqrt{q^2 k_x^2 |\varphi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right)}$$

III) Multi-scale shearing effects

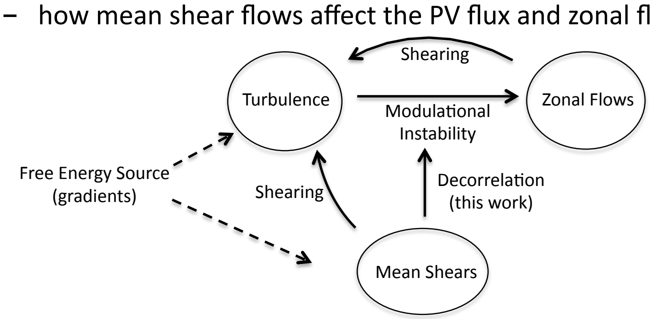
- Motivation
  - L-H transition intermediate phace: coexistence of mean shear and zonal flow



- Generic problem:
- coupling/interaction between different scale shearing fields



- Important issue
  - how mean shear flows affect the PV flux and zonal flow generation



Pei-Chun Hsu and P. H. Diamond, Phys. Plasmas, 22, 022306, (2015)

Modulational instab. w/ mean shear

Momentum flux -> Reynolds stress -> wave action $<\upsilon_x \upsilon_y> = -\sum_k \frac{k_x k_y}{k^4} N_k$

- mean shear in WKE: $\partial_t \tilde{N}_k + \upsilon_{gy}\partial_y \tilde{N}_k - k_x \langle V_x \rangle' \partial_{k_y} \tilde{N}_k + \delta\omega_k \tilde{N}_k = k_x \delta V_x' \partial_{k_y} N_0$

$$\upsilon_{gy} = \frac{2\beta k_x k_y}{k^4} \propto \Omega^{-3}$$

wave-packet excursion inhibited by mean shear

mean shear $-\partial_y \langle V_x \rangle = \Omega$

Non-linear diffusion

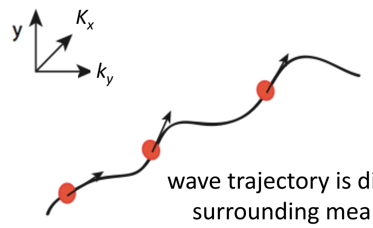
Seed ZF

- Ray trajectory refraction:

$$\frac{dk_y}{dt} = -\frac{\partial}{\partial y}(\omega + k_x V_x) \quad ; V_x = \langle V_x \rangle + \tilde{V}_x$$

$$\frac{dk_x}{dt} = 0$$

$$\frac{dy}{dt} = \upsilon_{gy} = \frac{2\beta k_x k_y}{k^4}$$



wave trajectory is distorted by surrounding mean shears

$$k_y(t) = k_y(0) + k_x \Omega t$$

$$y(t) = y(0) + e(t), \quad e(t) = \frac{\beta}{\Omega}\left(\frac{1}{k_0^2} - \frac{1}{k_x^2 + (k_{0y} + k_x \Omega t)^2}\right)$$

Modulational instab. w/ mean shear

- characteristic method (shearing frame)

original frame $\begin{pmatrix} k_y = k_y(0) + k_x \Omega t \\ y = y(0) + e(t) \end{pmatrix}$ $\quad \partial_t \tilde{N}_k + \upsilon_{gy}\partial_y \tilde{N}_k - k_x \langle V_x \rangle' \partial_{k_y} \tilde{N}_k + \delta\omega_k \tilde{N}_k = k_x \delta V_x' \partial_{k_y} N_0$

$\sim Dk^2$

cancel in shearing frame

shearing frame $\begin{pmatrix} \zeta = k_y - k_x \Omega t \\ \xi = y - e(t) \end{pmatrix}$ $\quad \partial_t G + D(k_x^2 + k_y^2) G = \delta(t - t_1)\delta(\xi - \xi_1)\delta(k_x - k_{1x})\delta(\zeta - \zeta_1)$

- Solving Green's function in shearing frame
- Changing variables back to original frame

--> $\tilde{N}_k(\Omega t >> 1) = \int_0^\infty d\tau e^{-\frac{\tau^3}{\tau_c^3} + i\Omega_q \tau + i\frac{qB}{k_0^2 \Omega}} k_x \delta\hat{V}' \partial_{k_y} N_0$

- strong mean shear limit ($\Omega >> \delta\omega_k$)

$$\tilde{N}_k \cong \left(\frac{3}{\delta\omega_k \Omega^2}\right)^{1/3} k_x \delta\hat{V}' \partial_{k_y} N_0 + O(\Omega^{-2})$$

$$\gamma_q, \langle \tilde{\upsilon}_y \nabla^2 \tilde{\psi} \rangle \propto \left(\frac{3}{\delta\omega_k \Omega^2}\right)^{1/3}$$

→ Mean shear reduces ZF growth ~$\Omega^{-2/3}$
→ scaling of PV flux in strong mean shear

Summary

- Inhomogeneous PV mixing is identified as the fundamental mechanism for zonal flow formation. *This study offered new perspectives and approaches to calculating spatial flux of PV.*

- The general structure of PV flux is studied by two non-perturbative relaxation models.

  1. selective decay model $\Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(-\frac{\langle q \rangle \partial_y \langle q \rangle}{\left(\partial_y \langle \phi \rangle\right)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$

  - PV flux contains diffusion and higher order diffusion terms. The homogenized quantity in the relaxed state is the ratio of PV gradient to zonal flow velocity. This is consistent with the structure of the PV staircase.

  2. PV-avalanche model $\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q$

  - PV flux is constrained by the joint reflection symmetry condition, and contains diffusive, hyper-diffusive, and convective terms. The convective transport of PV can be generalized to an effective diffusive transport.

Summary

- The transport coefficients are derived using perturbative analyses of wave kinetic equation. Relaxation models and perturbative analyses are *synergetic and complementary approaches.*

- In modulational instability analysis for a broadband spectrum, a negative viscosity and a positive hyper-viscosity, which represents ZF saturation mechanism, are derived. In parametric instability analysis for a narrow spectrum, a convective transport coefficient is obtained.

- Important issues addressed in our models includes PV staircase, turbulence spreading, avalanche-like transport, characteristic scales.

- The effect of the mean shear on PV flux and zonal flow formation is studied. ZF growth rate and the PV flux are shown to decreases with mean shearing rate as $\Omega^{-2/3}$. *Framework of PV transport for systems with multi-scale shearing fields is established.*

Extension of work

- What's the right relaxation principle?
  minimum enstrophy, maximun entropy, PV homogenization…
- Numerical simulation test:
  form of the PV flux during relaxation
  profile of the homogenized quantity for the relaxed state
  the minimum enstrophy in the relaxed state
  PV flux spectrum -1/f (?)
  staircase formation during relaxation
- Including the magnetic field
  β-plane MHD model of PV mixing processes (the effect of magnetic field on the cross phase of the Reynolds stress)
- Use PV flux expression to improve ZF dynamics representation in reduced L-H transition models

Onset of transition into H-mode

- Transition criterion

  turbulence power coupled to flow ≥ turbulence power increase

$$R_T = \frac{\langle \tilde{\upsilon}_x \tilde{\upsilon}_y \rangle \langle \upsilon_x \rangle'}{\langle \tilde{\upsilon}_\perp^2 \rangle (\gamma_{eff} - \gamma_{decorr})} \geq 1$$

G.R. Tynan et al 2013

criterion for turbulence collapse and transition onset
→ zonal flow role critical